Golod property of powers of ideals and of Koszul ideals

Rasoul Ahangari Maleki

Institute for Research in Fundamental Sciences (IPM)

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In this presentation all rings are commutative Noetherian local (or standard graded algebras over a field), and all modules are finitely generated (or graded finitely generated).

Rationality of Poincaré series and Golod rings

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$$\mathrm{P}_M^R(t) = f(t)/g(t)$$

for some complex polynomials $f(t)$ and $g(t)$.

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D. J. Anick (1980): The answer is negative.

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where $\mathbf{x} = x_1, \cdots, x_d$ is a minimal system of generators of $\mathfrak{m}$ and $\mathrm{H}_i(\mathbf{x};R)$ denotes the ith Koszul homology module of R w.r.t $\mathbf{x}$.

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Eagon resolution

Golod (1962): There exists a free resolution $\mathrm{E}_\bullet$ of k such that $\sum_{i\geq 0} rank(\mathrm{E}_i)t^i = \mathrm{Q}^R(t)$.

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Definition

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- $\mathsf{H}_+(\mathrm{K}_\bullet) = \oplus_{i=1} \mathsf{H}_i$ is a graded ideal of $\mathsf{H}(\mathrm{K}_\bullet)$ and is a k vector space of finite dimension.

Massey operations on the Koszul complex

We say all the Massey operations vanish (or $\mathrm{K}_\bullet$ admits a trivial Massey operation) if for some homogeneous k-basis $\mathcal{B} = \{h_\lambda\}_{\lambda \in \Lambda}$ of $\mathrm{H}_+(\mathrm{K}_\bullet)$ there exists a function $\mu : \bigcup_{n=1}^{d} \mathcal{B}^n \to \mathrm{K}_\bullet$, such that the following conditions are satisfied.

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Here if $x \in \mathrm{K}_i$ we set $\bar{x} = (-1)^{i+1}x$.

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Lescot (1990):

Let R be a Golod ring. Set $q_R(t) = 1 - t(\sum_{i=1}(\dim_k \mathrm{H}_i(\mathbf{x}; R)t^i)$. Then for any finitely generated R-module M one has.

$$q_R(t)\mathrm{P}_M^R(t) \in \mathbb{Z}[t].$$

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From now on $(S, \mathfrak{n}, k)$ is a regular local ring (or a polynomial ring over k)

A list of Golod ideals

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7. Herzog-Huneke (2013): If k has characteristic zero, then the ideals $\mathfrak{a}^m$, $\mathfrak{a}^{(m)}$ (the m-th symbolic power of $\mathfrak{a}$) and $\widetilde{\mathfrak{a}^m}$ (the saturated power of $\mathfrak{a}$) are Golod for all $m \geq 2$.

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We apply this and obtain new class of Golod ideals.

Let $\dim S = d$ and $\mathfrak{a}$ and $\mathfrak{b}$ be ideals (or graded ideals) of S. Let $K_\bullet$ be the Koszul complex of S w.r.t a minimal system of generators of the maximal ideal $\mathfrak{n}$. Denote by $Z_\bullet$ the cycles of $K_\bullet$.

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- Let $\rho_i(\mathfrak{b})$ be the smallest integer such that

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- Define

  $$\rho(\mathfrak{b}) = \max\{\rho_0(\mathfrak{b}), \cdots, \rho_{d-1}(\mathfrak{b})\}.$$
- $\rho(\mathfrak{b})$ is the smallest integer such that for all $m \geq \rho(\mathfrak{b})$ and $i \geq 1$, the maps

  $$\mathrm{Tor}_i^S(S/\mathfrak{b}^m, k) \to \mathrm{Tor}_i^S(S/\mathfrak{b}^{m-\rho(\mathfrak{b})}, k)$$

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3. $\mathfrak{a}$ is generated by a *part of a regular system of parameter* of S.

Question

Let S be a regular local ring. Is it true that $\rho(\mathfrak{a}) = 1$ for any proper ideal $\mathfrak{a}$ of S or equivalently that the map

$$\mathrm{Tor}_i^S(S/\mathfrak{a}^m, k) \to \mathrm{Tor}_i^S(S/\mathfrak{a}^{m-1}, k)$$

is zero for all $i > 0$ and all m?

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- Let $(R, \mathfrak{m}, k)$ be a local ring (or standard graded k-algebra) with the maximal (or homogeneous maximal) ideal $\mathfrak{m}$. An R-module M is called Koszul if its associated graded module $\operatorname{gr}_{\mathfrak{m}}(M) = \oplus_{i \geq 0} \mathfrak{m}^i M / \mathfrak{m}^{i+1} M$ as a graded $gr_{\mathfrak{m}}(R)$-module has a linear resolution.

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2. $\mathfrak{a}$ is a Koszul module;

Let S be a regular local ring and $\mathfrak{a}$ be a Koszul ideal of S. Is $\mathfrak{a}$ a Golod ideal?

Consider the map

$$\gamma_i(\mathfrak{b}) : \mathrm{Tor}_i^S(\mathfrak{n}\mathfrak{b}, k) \to \mathrm{Tor}_i^S(\mathfrak{b}, k)$$

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The following are equivalent.

1. $\mathfrak{a}$ is Koszul;
2. $\gamma_i^S(\mathfrak{n}^j\mathfrak{a}) = 0$ for all $i \geq 0$ and all $j \geq 0$.

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Let $(S, \mathfrak{n}, k)$ be a regular local ring and $\mathfrak{a}$ an ideal of S. Assume that the map

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Corollary

Let the situation be as above. If $\mathfrak{a}$ is a Koszul ideal, then $\mathfrak{a}$ is Golod.

Another characterization of Koszul ideals :

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$$\delta_i^j(\mathfrak{a}) : \mathrm{Tor}_i^S(\mathfrak{a}, S/\mathfrak{n}^{j+1}) \to \mathrm{Tor}_i^S(\mathfrak{a}, S/\mathfrak{n}^j)$$

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1 $\mathfrak{a}$ is Koszul;

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is zero for all $i > 0$, then $\mathfrak{a}\mathfrak{b}$ is *Golod* for any proper ideal $\mathfrak{b}$ with $\mathfrak{a} \subseteq \mathfrak{b}$.

Theorem

Let $(S, \mathfrak{n}, k)$ be a regular local ring and $\mathfrak{a}$ an ideal of S. Assume that the map

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Corollary

If $\mathfrak{a}$ is a *Koszul* ideal, then $\mathfrak{a}\mathfrak{b}$ is Golod for any proper ideal $\mathfrak{b}$ with $\mathfrak{a} \subseteq \mathfrak{b}$.

Thanks